Weighted Independent Sets in a Subclass of P_6-free Graphs

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Abstract

The *Maximum Weight Independent Set (MWIS)* problem on graphs with vertex weights asks for a set of pairwise nonadjacent vertices of maximum total weight. The complexity of the MWIS problem for P_6-free graphs is unknown.

In this note, we show that the MWIS problem can be solved in time $O(n^3m)$ for (P_6, banner)-free graphs by analyzing the structure of subclasses of these class of graphs. This extends the existing results for (P_5, banner)-free graphs, and (P_6, C_4)-free graphs. Here, P_t denotes the chordless path on t vertices, and a *banner* is the graph obtained from a chordless cycle on four vertices by adding a vertex that has exactly one neighbor on the cycle.

Keywords: Graph algorithms; Independent sets; P_6-free graphs.

1 Introduction

In an undirected graph G, an *independent set* is a set of mutually non-adjacent vertices. The Maximum Weight Independent Set (MWIS) problem asks for an independent set of maximum total weight in the given graph G with vertex weight function w on $V(G)$. The Maximum Independent Set (MIS) problem is the MWIS problem where all the vertices v in G have the same weight $w(v) = 1$. The MWIS problem on graphs ([GT20] in [13]) is one of the most investigated problems on graphs because of its applications in computer science, operations research, bioinformatics and other fields, including train dispatching [11] and data mining [26].

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If $\mathcal{F}$ is a family of graphs, a graph G is said to be *$\mathcal{F}$-free* if it contains no induced subgraph isomorphic to any graph in $\mathcal{F}$.

The MWIS problem is known to be NP-complete in general and hard to approximate; it remains NP-complete even on restricted classes of graphs such as triangle-free graphs [25], and ($K_{1,4}$,diamond)-free graphs [8]. Alekseev [1] showed that the M(W)IS problem remains NP-complete on H-free graphs, whenever H is connected, but neither a path nor a subdivision of the claw ($K_{1,3}$). On the other hand, the MWIS problem is known to be solvable in polynomial time on many graph classes, such as chordal graphs [12], perfect graphs [16], $2K_2$-free graphs [10], claw-free graphs [23], and fork-free graphs [22]. It is well known that the MWIS problem is solvable in linear time for the class of P_4-free graphs (also known as co-graphs) [9].

In this note, we focus on graphs which do not contain certain induced paths. As a natural generalization of P_4-free graphs, the class of P_k-free graphs ($k \geq 5$) has been studied widely in the literature. The complexity of the MWIS problem for P_5-free graphs was unknown for several decades. Recently, Lokshantov, Vatshelle and Villanger [20] showed an $O(n^{12}m)$ time algorithm for the MWIS problem on P_5-free graphs via minimal triangulations. However, the complexity of the MWIS problem is unknown for the class of P_6-free graphs. The MWIS problem is known to be solvable efficiently on several subclasses of P_k-free graphs ($k \geq 5$) by several techniques, and we refer to [3, 15, 18] and the references therein for a survey.

A vertex $z \in V(G)$ *distinguishes* two other vertices $x, y \in V(G)$ if z is adjacent to one of them and nonadjacent to the other. A vertex set $M \subseteq V(G)$ is a *module* in G if no vertex from $V(G) \setminus M$ distinguishes two vertices from M. The *trivial modules* in G are $V(G)$, $\emptyset$, and all one-elementary vertex sets. A graph G is *prime* if it contains only trivial modules. Note that prime graphs with at least three vertices are connected.

A class of graphs $\mathcal{G}$ is *hereditary* if every induced subgraph of a member of $\mathcal{G}$ is also in $\mathcal{G}$. We will use the following theorem by Lözin and Milanič [22].

Theorem 1 ([22]) *Let $\mathcal{G}$ be a hereditary class of graphs. If the MWIS problem can be solved in $O(n^p)$-time for prime graphs in $\mathcal{G}$, where $p \geq 1$ is a constant, then the MWIS problem can be solved for graphs in $\mathcal{G}$ in time $O(n^p + m)$.* ∎

A *clique* in G is a subset of pairwise adjacent vertices in G. A *clique separator/clique cutset* in a connected graph G is a subset Q of vertices in G such that Q is a clique and such that the graph induced by $V(G) \setminus Q$ is disconnected. A graph is an *atom* if it does not contain a clique separator.

Let $\mathcal{C}$ be a class of graphs. A graph G is *nearly* $\mathcal{C}$ if for every vertex v in $V(G)$ the graph induced by $V(G) \setminus N[v]$ is in $\mathcal{C}$. We will also use the following theorem given in [2]. Though the theorem (Theorem 1 of [2]) is stated only for hereditary class of graphs, the proof also work for any class of graphs, and is given below:

Theorem 2 ([2]) *Let $\mathcal{C}$ be a class of graphs such that MWIS can be solved in time $O(f(n))$ for every graph in $\mathcal{C}$ with n vertices. Then in any class of graphs whose atoms are all nearly $\mathcal{C}$ the MWIS problem can be solved in time $O(n \cdot f(n) + nm)$.* ■

We see that the Theorems 1 and 2 can be combined as follows:

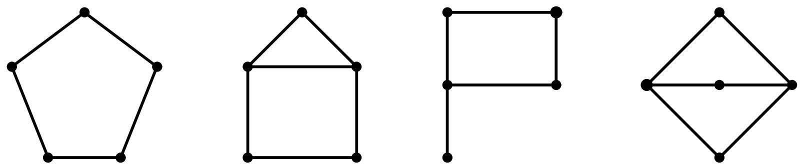
Theorem 3 *Let $\mathcal{G}$ be a hereditary class of graphs. Let $\mathcal{P}$ denotes the class of prime graphs in $\mathcal{G}$. Let $\mathcal{C}$ be a class of graphs such that MWIS can be solved in time $O(f(n))$ for every graph in $\mathcal{C}$ with n vertices. Suppose that every atom of a graph $G \in \mathcal{P}$ is nearly $\mathcal{C}$. Then the MWIS problem in $\mathcal{G}$ can be solved in time $O(n \cdot f(n) + nm)$.*

Proof. Let G be a graph in $\mathcal{G}$. First suppose that $G \in \mathcal{P}$. Since every atom of G is nearly $\mathcal{C}$, and since the MWIS problem for graphs in $\mathcal{C}$ can be solved in time $O(f(n))$, MWIS can be solved in time $O(n \cdot f(n) + nm)$ for G, by Theorem 2. Then the time complexity is the same when G is not prime, by Theorem 1. ■

In this note, using the above framework, we show that the MWIS problem in (P_6, banner)-free graphs can be solved in time $O(n^3 m)$, by analyzing the atomic structure and the MWIS problem in various subclasses of (P_6, banner)-free graphs, where a *banner* is the graph obtained from a chordless cycle on four vertices by adding a vertex that has exactly one neighbor on the cycle (see also Figure 1). This result extends the results known for P_4-free graphs, (P_6, C_4)-free graphs, and for (P_5, banner)-free graphs [3, 21].

We note that applying Corollary 9 in [3] which used an approach for solving MWIS by combining prime graphs and atoms, it was claimed in [5] that MWIS is solvable efficiently in time $O(n^7 m)$ for (P_6, banner)-free graphs. However, Corollary 9 in [3] is not proven (and thus has to be avoided).

It is also noteworthy that the MWIS problem remains NP-complete in banner-free graphs (this follows from the result of Murphy [24] for graphs with large girth). The class of banner-free graphs is of particular interest, since it contains two important subclasses where the MWIS problem can



C_5 House Banner $K_{2,3}$

Figure 1: Some special graphs

be solved efficiently, namely claw-free graphs and P_4-free graphs. Also, the complexity of the MIS Problem for various subclasses of banner-free graphs has been studied in the literature, and we refer to [14, 21] for more details.

2 Notation and Terminology

For notation and terminology not defined here, we follow [6]. Let G be a finite, undirected, simple graph with vertex-set $V(G)$ and edge-set $E(G)$. We let $|V(G)| = n$ and $|E(G)| = m$. The symbols P_k and C_k respectively denotes the chordless path and the chordless cycle on k vertices. Let $K_{m,n}$ denote the complete bipartite graph with m vertices in one partition set and n vertices in the other. The *banner* is also called P or 4-*apple* or A_4 in various papers [3, 5, 7], and see Figure 1 for some of the special graphs that we have used in this paper.

For a vertex $v \in V(G)$, the *neighborhood* $N(v)$ of v is the set $\{u \in V(G) \mid uv \in E(G)\}$, and its *closed neighborhood* $N[v]$ is the set $N(v) \cup \{v\}$. The neighborhood $N(X)$ of a subset $X \subseteq V(G)$ is the set $\{u \in V(G) \setminus X \mid u$ is adjacent to a vertex of $X\}$, and its closed neighborhood $N[X]$ is the set $N(X) \cup X$. Let $\overline{N[X]}$ denote the set $V(G) \setminus N[X]$. Given a subgraph H of G and $v \in V(G) \setminus V(H)$, let $N_H(v)$ denote the set $N(v) \cap V(H)$, and for $X \subseteq V(G) \setminus V(H)$, let $N_H(X)$ denote the set $N(X) \cap V(H)$. For any two subsets S, $T \subseteq V(G)$, we say that S is *complete* to T if every vertex in S is adjacent to every vertex in T.

The following notation will be used several times in the proofs. Given a graph G, let v be a vertex in G and H be an induced subgraph of $G \setminus N[v]$. Let $t = |V(H)|$. Then we define the following sets:

$$\begin{aligned}
Q &= \text{the component of } G \setminus N[H] \text{ that contains } v,\\
A_i &= \{x \in V(G) \setminus V(H) \mid |N_H(x)| = i\}\ (1 \le i \le t),\\
A_i^+ &= \{x \in A_i \mid N(x) \cap Q \neq \emptyset\},
\end{aligned}$$

$$
\begin{aligned}
A_i^- &= \{x \in A_i \mid N(x) \cap Q = \emptyset\}, \\
A^+ &= A_1^+ \cup \cdots \cup A_t^+ \text{ and } A^- = A_1^- \cup \cdots \cup A_t^-.
\end{aligned}
$$

So, $N(H) = A^+ \cup A^-$. Note that, by the definition of Q and A^+, we have $A^+ = N(Q)$. Hence A^+ is a separator between H and Q in G.

3 MWIS in (P_6, banner)-free graphs

In this section, we prove that the MWIS problem in (P_6, banner)-free graphs can be solved in $O(n^3m)$-time, by analyzing the atomic structure and the MWIS problem in various subclasses of (P_6, banner)-free graphs.

3.1 MWIS in (P_6, C_4)-free graphs

Theorem 4 *The MWIS problem can be solved in time $O(nm)$ for (P_6, C_4)-free graphs.*

Proof. In [3], Brandstädt and Hoáng showed that atoms of (P_6, C_4)-free graphs are either nearly chordal or 2-specific graphs (see [3] for the definition of 2-specific graphs). Since the MWIS problem is trivial for 2-specific graphs, and can be solved in time $O(m)$ for chordal graphs [12], by Theorem 2, the MWIS problem for (P_6, C_4)-free graphs can be solved in time $O(nm)$. ∎

3.2 MWIS in (P_6, banner, house)-free graphs

In this section, we will show that the MWIS can be solved in $O(nm)$-time for (P_6, banner, house)-free graphs. Though the following lemma can be derived from a result of Hoáng and Reed [17], we give a simple proof here for completeness.

Lemma 3.1 *If $G = (V, E)$ is a prime (banner, house)-free graph, then G is C_4-free.*

Proof. Suppose to the contrary that G contains an induced C_4 with vertex set $\{a_1, a_2, b_1, b_2\}$ and edge set $\{a_1b_1, b_1a_2, a_2b_2, b_2a_1\}$. Let Q be the connected component in the complement of the graph $G[N(a_1) \cap N(a_2)]$ that contains b_1 and b_2. Since G is prime, there exist vertices $b_1', b_2' \in Q$ such that $b_1'b_2' \notin E$, which are distinguished by a vertex $z \notin Q$, say $zb_1' \in E$ and $zb_2' \notin E$. Then since $\{z, b_1', b_2', a_1, a_2\}$ does not induce a house or a banner in G, we have $za_1 \in E$ and $za_2 \in E$. Hence $z \in Q$, which is a contradiction. This shows Lemma 3.1. ∎

Then we immediately have the following:

Theorem 5 *The MWIS problem can be solved in $O(nm)$-time for (P_6, banner, house)-free graphs.*

Proof. Let G be a prime (P_6, banner, house)-free graph. Then by Lemma 3.1, G is (P_6, C_4)-free. Since the MWIS problem can be solved in time $O(nm)$ for (P_6, C_4)-free graphs (by Theorem 4), the MWIS problem can be solved in time $O(nm)$ for prime (P_6, banner, house)-free graphs. Then the time complexity is same when G is not prime, by Theorem 1. ■

3.3 MWIS problem in (P_6, banner, C_5)-free graphs

In this section, we show that the MWIS problem can be solved in time $O(n^2m)$ for (P_6, banner, C_5)-free graphs. We use the following lemma given by Brandstädt et al. in [5].

Lemma 3.2 ([5]) *(see also [7]) Prime banner-free graphs are $K_{2,3}$-free.* ■

Theorem 6 *Let G be a prime (P_6, banner, C_5)-free graph. Then every atom of G is nearly house-free.*

Proof. Let G' be an atom of G. We want to show that G' is nearly house-free, so let us assume on the contrary that there is a vertex $v \in V(G')$ such that $G' \setminus N[v]$ contains an induced house H. Let H have vertex set $\{v_1, v_2, v_3, v_4, v_5\}$ and edge set $\{v_1v_2, v_2v_3, v_3v_4, v_4v_1, v_2v_5, v_3v_5\}$. For $i = 1, 2, \ldots, 5$, we define sets A_i, A_i^+, A_i^-, A^+, A^-, and Q as in the last paragraph of Section 2.

Note that by the definition of Q and A^+, we have $A^+ = N(Q)$. Hence A^+ is a separator between H and Q in G. Now we have the following:

Claim 6.1 *$A_1^+ = \emptyset$.*

Proof of Claim 6.1. Suppose not, and let $x \in A_1^+$. Then by the definition of A_1^+, there exists a vertex $y \in Q$ such that $xy \in E$. Now:

(i) If $N_H(x) = \{v_1\}$ or $\{v_2\}$, then $\{v_1, v_2, v_3, v_4, x\}$ induces a banner in G, which is a contradiction.

(ii) If $N_H(x) = \{v_5\}$, then $\{y, x, v_5, v_3, v_4, v_1\}$ induces a P_6 in G, which is a contradiction.

Since the other cases are symmetric, Claim 6.1 is proved. ♦

Claim 6.2 *If $x \in A_2^+$, then $N_H(x)$ is either $\{v_2, v_3\}$ or $\{v_1, v_4\}$.*

Proof of Claim 6.2. Suppose not. Since $x \in A_2^+$, there exists a vertex $y \in Q$ such that $xy \in E$. Now:

(i) If $N_H(x) = \{v_1, v_2\}$, then $\{y, x, v_1, v_4, v_3, v_5\}$ induces a P_6 in G, which is a contradiction.

(ii) If $N_H(x) = \{v_2, v_4\}$, then $\{y, x, v_2, v_3, v_4\}$ induces a banner in G, which is a contradiction.

(iii) If $N_H(x) = \{v_1, v_5\}$, then $\{v_1, v_2, v_5, x, y\}$ induces a banner in G, which is a contradiction.

(iv) If $N_H(x) = \{v_2, v_5\}$, then $\{y, x, v_5, v_3, v_4, v_1\}$ induces a P_6 in G, which is a contradiction.

Since the other cases are symmetric, Claim 6.2 is proved. ♦

By Claim 6.2, we define sets $B_1 = \{x \in A_2^+ \mid N_H(x) = \{v_2, v_3\}\}$ and $B_2 = \{x \in A_2^+ \mid N_H(x) = \{v_1, v_4\}\}$. Then $A_2^+ = B_1 \cup B_2$.

Claim 6.3 *If $B_1 \neq \emptyset$, then $B_2 = \emptyset$, and vice versa.*

Proof of Claim 6.3. Assume the contrary, and let $x \in B_1$ and $y \in B_2$. Then since $\{x, y, v_1, v_2, v_5\}$ does not induce a banner in G, $xy \notin E$. Since $x, y \in A_2^+$ and Q is connected, there exists a path $z_0 - \cdots - z_p$ inside Q such that $xz_0 \in E$ and $yz_p \in E$, and we choose a shortest such path. If $p = 0$, then $\{z_0, x, v_2, v_1, y\}$ induces a C_5 in G, which is a contradiction. So, $p \geq 1$. But, then $\{z_1, z_0, x, v_2, v_1, v_4\}$ induces a P_6 in G, which is a contradiction. So the claim holds. ♦

Claim 6.4 *If $x \in A_3^+$, then $N_H(x) = \{v_2, v_3, v_5\}$.*

Proof of Claim 6.4. Suppose not. Since $x \in A_3^+$, there exists a vertex $y \in Q$ such that $xy \in E$. Now:

(i) If $N_H(x) = \{v_1, v_2, v_3\}$, then $\{y, x, v_1, v_4, v_3\}$ induces a banner in G, which is a contradiction.

(ii) If $N_H(x) = \{v_1, v_2, v_4\}$, then $\{y, x, v_2, v_3, v_4\}$ induces a banner in G, which is a contradiction.

(iii) if $N_H(x) = \{v_1, v_4, v_5\}$ or $\{v_1, v_3, v_5\}$, then $\{y, x, v_1, v_2, v_5\}$ induces a banner in G, which is a contradiction.

(iv) if $N_H(x) = \{v_1, v_2, v_5\}$, then $\{x, v_1, v_4, v_3, v_5\}$ induces a C_5 in G, which is a contradiction.

Since the other cases are symmetric, Claim 6.4 is proved. ♦

Claim 6.5 *If $x \in A_4^+$, then $N_H(x) = \{v_1, v_2, v_3, v_4\}$.*

Proof of Claim 6.5. Suppose not. Since $x \in A_4^+$, there exists a vertex $y \in Q$ such that $xy \in E$. Now:

(i) If $N_H(x) = \{v_1, v_2, v_3, v_5\}$, then $\{y, x, v_1, v_4, v_3\}$ induces a banner in G, which is a contradiction.

(ii) if $N_H(x) = \{v_1, v_2, v_4, v_5\}$, then $\{y, x, v_5, v_3, v_4\}$ induces a banner in G, which is a contradiction.

Since the other cases are symmetric, the claim holds. ♦

By Claim 6.1, we have $A^+ = A_2^+ \cup A_3^+ \cup A_4^+ \cup A_5^+$.

Claim 6.6 *$A^+ \setminus B_2$ is a clique.*

Proof of Claim 6.6. Suppose to the contrary that there are non-adjacent vertices $x, y \in A^+ \setminus B_2$. Then by Claims 6.1, 6.2, 6.4, and 6.5, and by the definition of A_5^+, we have $\{v_2, v_3\} \subseteq N_H(x) \cap N_H(y)$. Since $x, y \in A^+ \setminus B_2$ and Q is connected, there exists a path $z_0 - \cdots - z_p$ inside Q such that $xz_0 \in E$ and $yz_p \in E$, and we choose a shortest such path. Suppose that $p = 0$. We claim that there is no edge between $\{v_1, v_4\}$ and $\{x, y\}$. For suppose on the contrary and without loss of generality that $v_1x \in E$. Then $\{x, y, z_0, v_1, v_3\}$ induces either a $K_{2,3}$ in G (if $v_1y \in E$), which contradicts Lemma 3.2, or a banner in G (if $v_1y \notin E$), which is a contradiction. So the claim holds. But, then $\{z_0, x, y, v_2, v_1\}$ induces a banner in G, which is a contradiction. Hence $p \geq 1$. If $p = 1$, then $\{v_2, x, z_0, z_1, y\}$ induces a C_5 in G, a contradiction. So, suppose that $p \geq 2$. Then since $\{z_2, z_1, z_0, x, v_3, v_4\}$ does not induce a P_6 in G, $xv_4 \in E$. Again, since $\{z_0, z_1, z_2, y, v_3, v_4\}$ does not induce a P_6 in G, $yv_4 \in E$. But, then $\{z_0, x, y, v_2, v_4\}$ induces a banner in G, which is a contradiction. Thus, Claim 6.6 is proved. ♦

Claim 6.7 *B_2 is a clique.*

Proof of Claim 6.7. Suppose to the contrary that there are non-adjacent vertices $x, y \in B_2$. Since Q is connected, there exists a path $z_0 - \cdots - z_p$ inside Q such that $xz_0 \in E$ and $yz_p \in E$, and we choose a shortest such path. Now, $\{z_0, x, y, v_1, v_2\}$ induces a banner in G (if $p = 0$), and $\{z_1, z_0, x, v_4, v_3, v_5\}$ induces a P_6 in G (if $p \geq 1$), a contradiction. So the claim holds. ♦

Claim 6.8 *A^+ is a clique.*

Proof of Claim 6.8. Suppose to the contrary that there are non-adjacent vertices $x, y \in A^+$. By Claims 6.6 and 6.7, we may assume that $x \in A^+ \setminus B_2$ and $y \in B_2$. Since Q is connected, there exists a path $z_0 - \cdots - z_p$ inside Q such that $xz_0 \in E$ and $yz_p \in E$, and we choose a shortest such path. Now, if $p \geq 2$, then $\{y, z_p, \ldots, z_0, x, v_3\}$ induces a path P_t ($t \geq 6$) in G, a contradiction. So, $p \leq 1$. Suppose that $p = 0$. Since $\{v_4, x, y, z_0, v_2\}$ does not induce a banner in G, $v_4x \notin E$. But now, $\{z_0, x, y, v_3, v_4\}$ induces a C_5 in G, which is a contradiction. So, $p = 1$. Then $\{v_4, y, z_1, z_0, x\}$ induces a C_5 in G (if $xv_4 \in E$), and $\{v_4, y, z_1, z_0, x, v_2\}$ induces a P_6 in G (if $xv_4 \notin E$), a contradiction. So the claim holds. ♦

Since A^+ is a separator between H and Q in G, we obtain that $V(G') \cap A^+$ is a clique separator in G' between H and $V(G') \cap Q$ (which contains v). This is a contradiction to the fact that G' is an atom. This proves Theorem 6. ■

Using Theorem 6, we now prove the following:

Theorem 7 *The MWIS problem can be solved in $O(n^2m)$-time for (P_6, banner, C_5)-free graphs.*

Proof. Let G be a (P_6, banner, C_5)-free graph. First suppose that G is prime. By Theorem 6, every atom of G is nearly house-free. Since the MWIS problem in (P_6, banner, house)-free graphs can be solved in time $O(nm)$ (by Theorem 5), MWIS can be solved in time $O(n^2m)$ for G, by Theorem 2. Then the time complexity is the same when G is not prime, by Theorem 1. ■

3.4 MWIS problem in (P_6, banner)-free graphs

In this section, we show that the MWIS problem can be solved in time $O(n^3m)$ for (P_6, banner)-free graphs. In [5], it was shown that prime atoms of (P_6, banner)-free graphs are nearly C_5-free. Applying Corollary 9 in [3] which used an approach for solving MWIS by combining prime graphs and

atoms, it was claimed in [5] that MWIS is solvable efficiently in time $O(n^7m)$ for (P_6, banner)-free graphs . However, Corollary 9 in [3] is not proven (and thus has to be avoided); a correct way would be to show that atoms of prime (P_6, banner)-free graphs are nearly C_5-free (see also [4, 19] for examples). This will be done in the proof of Theorem 8. Though the proof given here is very similar to that of [5], here we carefully analyze and reprove it so as to apply the known theorems stated in Section 1.

Theorem 8 *Let G be a prime (P_6, banner)-free graph. Then every atom of G is nearly C_5-free.*

Proof. Let G' be an atom of G. We want to show that G' is nearly C_5-free, so let us assume on the contrary that there is a vertex $v \in V(G')$ such that $G' \setminus N[v]$ contains an induced C_5, say H with vertices $\{v_1, \ldots, v_5\}$ and edges v_iv_{i+1}, for $i \in \{1, \ldots, 5\}$ (i mod 5). For $i = 1, \ldots, 5$ we define sets A_i, A_i^+, A_i^-, A^+, A^-, and Q as in the last paragraph of Section 2.

Note that by the definition of Q and A^+, we have $A^+ = N(Q)$. Hence A^+ is a separator between H and Q in G. Throughout this proof, we take all the subscripts of v_i to be modulo 5. Then we have the following:

Since G is (P_6, banner)-free, it is easy to see that $A_1^+ \cup A_2^+ \cup A_4^+ = \emptyset$. So, $A^+ = A_3^+ \cup A_5^+$.

Claim 8.1 *If $x \in A_3$, then $N_H(x) = \{v_{i-1}, v_i, v_{i+1}\}$, for some $i \in \{1, \ldots, 5\}$.*

Proof of Claim 8.1. Suppose not. Up to symmetry and without loss of generality, we may assume that $\{v_i, v_{i+2}\} \subseteq N_H(x)$. Now, $\{y, x, v_i, v_{i+1}, v_{i+2}\}$ induces a banner in G, which is a contradiction. Hence the claim. ♦

By Claim 8.1, define sets $D_i = \{x \in A_3^+ \mid N_H(x) = \{v_{i-1}, v_i, v_{i+1}\}\}$, for $i \in \{1, \ldots, 5\}$ (i mod 5). So, $A_3^+ = \cup_{i=1}^5 D_i$. Now we prove the following:

Claim 8.2 *A_3^+ is a clique.*

Proof of Claim 8.2. Suppose to the contrary that there are non-adjacent vertices $x, y \in A_3^+$. Since $x \in A_3^+$, there exists a vertex $z \in Q$ such that $xz \in E$. Also, $x \in D_i$ and $y \in D_j$, for some i and j, where $i, j \in \{1, \ldots, 5\}$. Now:

(i) If $x, y \in D_1$, then $\{x, y, v_2, v_3, v_5\}$ induces a banner in G, which is a contradiction.

(ii) If $x \in D_1$ and $y \in D_2$, then since $\{z, x, v_5, v_4, v_3, y\}$ does not induce a P_6 in G, $yz \in E$. But, then $\{z, x, v_2, y, v_5\}$ induces a banner in G, which is a contradiction.

(iii) If $x \in D_1$ and $y \in D_3$, then since $\{z, x, v_2, y, v_4\}$ does not induce a banner in G, $yz \notin E$. Since $y \in A_3^+$, there exists a vertex $z' \in Q$ such that $yz' \in E$. Again, since $\{z', x, v_2, y, v_4\}$ does not induce a banner in G, $xz' \notin E(G)$. Then since $\{z, x, v_5, v_4, y, z'\}$ does not induce a P_6 in G, $zz' \in E(G)$. But, now $\{v_1, x, z, z', y, v_4\}$ induces P_6 in G, which is a contradiction.

Since the other cases are symmetric, Claim 8.2 is proved. ♦

Claim 8.3 *A_5^+ is a clique.*

Proof of Claim 8.3. Suppose to the contrary that there are non-adjacent vertices $x, y \in A_5^+$. Since $x \in A_5^+$, there exists a vertex $z \in Q$ such that $xz \in E$. Then since $\{v_1, v_3, x, y, z\}$ does not induce a banner in G, $yz \in E$. But, then $\{v_1, v_3, x, y, z\}$ induces a $K_{2,3}$ in G, which is a contradiction to Lemma 3.2. Thus, Claim 8.3 is proved. ♦

Claim 8.4 *A_3^+ and A_5^+ are complete to each-other.*

Proof of Claim 8.4. Suppose to the contrary that there are non-adjacent vertices $x \in A_3^+$ and $y \in A_5^+$. Up to symmetry, by Claim 8.1, we may assume that $N_H(x) = \{v_1, v_2, v_3\}$. Since $y \in A_5^+$, there exists a vertex $z \in Q$ such that $yz \in E$. Then since $\{x, v_1, v_3, y, z\}$ does not induce a banner in G, $xz \in E$. But, then $\{z, x, v_1, y, v_4\}$ induces a banner in G, which is a contradiction. So the claim holds. ♦

Since $A^+ = A_3^+ \cup A_5^+$, and by Claims 8.2, 8.3 and 8.4, we see that A^+ is a clique. Since A^+ is a separator between H and Q in G, we obtain that $V(G') \cap A^+$ is a clique separator in G' between H and $V(G') \cap Q$ (which contains v). This is a contradiction to the fact that G' is an atom. This proves Theorem 8. □

Using Theorem 8, we now prove the following:

Theorem 9 *The MWIS problem can be solved in $O(n^3m)$-time for (P_6, banner)-free graphs.*

Proof. Let G be an (P_6, banner)-free graph. First suppose that G is prime. By Theorem 8, every atom of G is nearly C_5-free. Since the MWIS problem can be solved in time $O(n^2m)$ for (P_6, banner, C_5)-free graphs (by Theorem 7), MWIS can be solved in time $O(n^3m)$ for G, by Theorem 2. Then the time complexity is the same when G is not prime, by Theorem 1. ■

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